View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Virtual Commons - Bridgewater State University

Bridgewater State University
Virtual Commons - Bridgewater State University

Mathematics Faculty Publications

Mathematics Department

2013

Concatenation of Consecutive Fibonacci and Lucas Numbers: a Lesson in Patterns of Divisibility and Proof

Thomas Moore
Bridgewater State University, MOORE@bridgew.edu

Virtual Commons Citation

Moore, Thomas (2013). Concatenation of Consecutive Fibonacci and Lucas Numbers: a Lesson in Patterns of Divisibility and Proof. In *Mathematics Faculty Publications.* Paper 34.
Available at: http://vc.bridgew.edu/math_fac/34

This item is available as part of Virtual Commons, the open-access institutional repository of Bridgewater State University, Bridgewater, Massachusetts.

Concatenation of consecutive Fibonacci and Lucas Numbers: a lesson in patterns of divisibility and proof by induction

Tom Moore

Professor Emeritus of Mathematics

Bridgewater State University

Bridgewater, MA

02325

June 26, 2013

Abstract: the lure of the Fibonacci sequence and the related Lucas sequence for those who seek numerical patterns is well known. In this note we discover that concatenating consecutive terms in both sequences leads to observations on the results being divisible by certain positive integers.

The Fibonacci sequence F_n begins 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … and is governed by the recurrence relation $F_n = F_{n-1} + F_{n-2}, n \geq 3$ with $F_1 = 1, F_2 = 1$. Fibonacci (ca. 1175-1250), also known as Leonardo of Pisa, wrote a book, the *Liber Abaci*, around 1202, which has a significant place in mathematical history. In it he advocated for the use of the Hindu-Arabic decimal system (base 10) over the use of Roman numeration. And, also in the book, is a question whose answer introduces the famous sequence described above.

The sequence is a treasure trove of numerical patterns. For example, one notices $F_5 = 5, F_{10} = 55$ and $F_{15} = 610$ are each divisible by 5. These particular numbers are visibly multiples of 5 because they have units' digit 5 or 0. However, to see that *every* fifth Fibonacci number is divisible by 5, we must produce a proof and, in our case, a proof by mathematical induction.

We claim then that F_{5k} is divisible by 5, for all positive integers $k \geq 1$.

As shown above, this is true for $k = 1, 2, 3$. This begins an induction argument. Assume

$F_{5k} = 5t$, for some positive integers k and t. We will deduce it is true for $F_{5(k+1)} = F_{5k+5}$.

To see this we decompose terms utilizing the Fibonacci recurrence relation and, of course, we use our assumption that $F_{5k} = 5t$:

$$
\begin{aligned}
F_{5k+5} &= F_{5k+4} + F_{5k+3} \\
&= (F_{5k+3} + F_{5k+2}) + (F_{5k+2} + F_{5k+1}) \\
&= (F_{5k+2} + F_{5k+1}) + (F_{5k} + F_{5k+1}) + (F_{5k} + F_{5k+1}) + F_{5k+1} \\
&= (F_{5k+1} + F_{5k}) + 4F_{5k+1} + 2F_{5k} \\
&= 5F_{5k+1} + 3F_{5k} \\
&= 5F_{5k+3} + 3(5t) \\
&= 5(F_{5k+3} + 3t), \text{ as desired.}
\end{aligned}
$$

Now, given two numbers, a and b, we may add, subtract, multiply and divide them as usual. However we may perform other operations "$*$" on them such as exponentiation: $a * b = a^b$. In this paper we look at the operation of concatenation of two positive integers x and y, in that order, which simply places x in front of y. We will write this operation as $x \sim y$.

For example, $53 \sim 126 = 53126$. Notice that this concatenation is equivalent to

$53(10^3) + 126 = 53126$ and, likewise, $146 \sim 37 = 14637$ is equivalent to $146(10^2) + 37 = 14637$.

The result proved above, that F_{5k} is always a multiple of 5, implies that, if we prefix F_{5k} using any of the base 10 digits 0,1,…, 9 , as many times as we want, then the resulting number is again a multiple of 5. In particular, we immediately have the kind of result we are looking for in this paper, that $F_{5k-1} \sim F_{5k}$ is divisible by 5, $\forall k \geq 1$.

Examples. Among these concatenated Fibonacci numbers the first few are 35, 3455 and 377610.

Now this concatenation result is a bit underwhelming, given that it is encompassed by the ability to prefix any digits at all in front of F_{5k} . So let us turn to a more interesting result. First, we observe that $F_4 = 3, F_8 = 21 \text{and } F_{12} = 144$ and that all these numbers are multiples of 3. Recall that there is a well known digit test (base 10) for divisibility by 3, namely, a given number *n* is divisible by 3 if and only if the number formed by the sum of the digits of *n* is divisible by 3. For example, 12234072 is divisible by 3 by this criterion. However, to see that *every* fourth Fibonacci number is divisible by 3, we again turn to mathematical induction.

Our claim is that F_{4k} is divisible by 3, $\forall k \geq 1$.

The above examples show that this is true for $k = 1, 2, 3$. Now assume that F_{4k} is divisible by 3, for some k, say $F_{4k} = 3t$. We must deduce the same is true for $F_{4(k+1)} = F_{4k+4}$.

$$
\begin{aligned}
F_{4k+4} &= F_{4k+2} + F_{4k+3} \\
&= (F_{4k} + F_{4k+1}) + (F_{4k+1} + F_{4k+2}) \\
&= F_{4k} + 2F_{4k+1} + (F_{4k} + F_{4k+1}) \\
&= 3F_{4k+1} + 2F_{4k} \\
&= 3F_{4k+1} + 2(3t) \\
&= 3(F_{4k+1} + 2t), \text{ as desired.}
\end{aligned}
$$

Now, prefixing any digits before F_{4k} will not necessarily produce a number divisible by 3. For example 21 is a multiple of 3 but 421 is not. But, examining the Fibonacci numbers and concatenating consecutive terms, we discover that $F_2 \sim F_3 = 12 = (3)(4)$ and $F_6 \sim F_7 = 813 = (3)(271)$ and $F_{10} \sim F_{11} = 5589 = (3)(1863)$ .This is suggestive and, indeed, we are able to prove that a concatenation of certain consecutive Fibonacci numbers has the desired property in our next result. In its proof we employ the notation $\equiv_3$ standing for "is congruent mod 3 to". By definition, $a \equiv_3 b$ means a and b have the same remainder when divided by 3. For example, 10 $\equiv_3$ 1, so also $10^q \equiv_3 1$ for any positive integer q.

We are claiming that $F_{4k+2} \sim F_{4k+3}$ is divisible by 3, $\forall k \geq 1$.

This follows directly from the previous result on F_{4k+4} like this:

$$\begin{aligned}
F_{4k+2} \sim F_{4k+3} &= F_{4k+2}(10^q) + F_{4k+3}, \text{ for some positive integer } q, \\
&\equiv_3 F_{4k+2}(1) + F_{4k+3} \\
&\equiv_3 F_{4k+2} + F_{4k+3} \\
&\equiv_3 F_{4k+4}
\end{aligned}$$

and, since this last number is a multiple of 3, so is the number $F_{4k+2} \sim F_{4k+3}$.

The Lucas sequence L_n begins 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, … and is governed by the same recurrence relation as the Fibonacci numbers, but with different start values. Namely, $L_n = L_{n-1} + L_{n-2}, n \geq 3$, with $L_1 = 2, L_2 = 1$. Edouard Lucas (1842-1891) was a French mathematician perhaps best known for his invention of the *Tower of Hanoi* game in which disks of different sizes are stacked on one of three pegs and must be restacked one disk at a time on another peg while never placing a larger disk atop a smaller one.

As before, we will consider the concatenation $L_n \sim L_{n+1}$ of consecutive Lucas numbers and test the result for divisibility by certain numbers, namely, 3 and 9.

We observe

$$\begin{aligned}
L_1 \sim L_2 &= 21 = (3)(7) \\
L_5 \sim L_6 &= 711 = (9)(79) \\
L_9 \sim L_{10} &= 4776 = (3)(1592)
\end{aligned}$$

and also that 3 divides $L_3 = 3$; 3 divides $L_7 = 18$; and 3 divides $L_{11} = 123$.

These suggest firstly that $L_{4k+1} \sim L_{4k+2}$ is a multiple of 3. And secondly, that the theorem to which our desired concatenation is a consequence is this

L_{4k+3} is divisible by 3, $\forall k \geq 0$.

Let us prove this last statement. If k = 0, 1, or 2 this is true by the above examples. Now assume that L_{4k+3} is divisible by 3 for some k, say $L_{4k+3} = 3t$. We must deduce that $L_{4(k+1)+3} = L_{4k+7}$ is divisible by 3.

$$\begin{aligned}
L_{4k+7} &= L_{4k+5} + L_{4k+6} \\
&= (L_{4k+3} + L_{4k+4}) + (L_{4k+4} + L_{4k+5}) \\
&= L_{4k+3} + L_{4k+4} + L_{4k+4} + (L_{4k+3} + L_{4k+4}) \\
&= 2L_{4k+3} + 3L_{4k+4} \\
&= 2(3t) + 3L_{4k+4} \\
&= 3(2t + L_{4k+4}), \text{ as desired.}
\end{aligned}$$

And now we may deduce the concatenation result: $L_{4k+1} \sim L_{4k+2}$ is divisible by 3, $\forall k \geq 0$. For

$$
\begin{aligned}
L_{4k+1} \sim L_{4k+2} &= L_{4k+1}(10^q) + L_{4k+2}, \text{ for some positive integer } q, \\
&\equiv_3 L_{4k+1}(1) + L_{4k+2} \\
&\equiv_3 L_{4k+3} \\
&\equiv_3 0, \text{ by the previous result.}
\end{aligned}
$$

We leave the proofs of the next result and its corollary to the interested reader. Be assured that these proofs may be constructed along the lines of what we have done in each case above.

Notice that 9 divides each of $\mathrm{L}_7 = 18, L_{19} = 5778 \text{ and } L_{31} = 1860418$. This is easily seen via the digit divisibility test for 9, analogous to that for 3, namely, a given number *n* is divisible by 9 if and only if the number formed by the sum of the digits of *n* is divisible by 9. But to show that *every* L_{12k+7} is divisible by 9 will require an inductive proof (again, left to the interested reader).

And so, L_{12k+7} is divisible by 9, $\forall k \geq 0$.

Our concatenation corollary follows, easily proved in the style of the previous ones. That is,

$L_{12k+5} \sim L_{12k+6}$ is divisible by 9, $\forall k \geq 0$.

For example, with $k = 0$ we have $L_5 \sim L_6$ =711 with sum of digits 9, and with k =1 we have

$L_{17} \sim L_{18} = 22073571 \text{ with sum of digits } 27$.

Summary. Our look at two special sequences and the concatenation of consecutive terms in them has paid off in patterns of division and in providing the need for proofs by induction. Other special sequences can be looked at in this way. For instance the author has considered the sequence of triangular numbers in this same way. This led to a problem which will be posed in an upcoming issue of *Mathematical Spectrum* (a journal of The Applied Probability Trust in the UK).

Moreover, the novel idea of studying concatenation of integers allows the teacher and students a new and perhaps exciting chance at pattern discovery, reasoning and proof. This fits into the curriculum framework as noted in *The Ontario Curriculum (Mathematics) Grades 9 and 10*, page 13, Reasoning and Proving: An emphasis on reasoning helps students make sense of mathematics. Classroom instruction in mathematics should always foster critical thinking – that is, an organized, analytical,well-reasoned approach to learning mathematical concepts and processes and to solving problems.

As students investigate and make conjectures about mathematical concepts and relationships, they learn to employ *inductive reasoning*, making generalizations based on specific findings from their investigations. Students also learn to use counter-examples to disprove conjectures. Students can use *deductive reasoning* to assess the validity of conjectures and to formulate proofs.